MATH 564 2023/09/19
Advanced Real Analysis

Abstract Pocket Tools

Let $(X, \mathcal{B}, \mu)$ be a measure space.

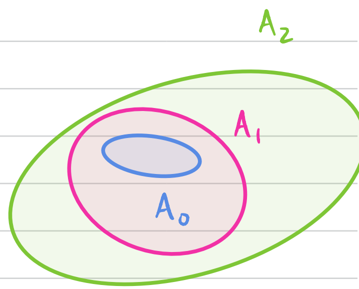
Monotone convergence for sets

a) If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence of sets in $\mathcal{B}$, ie

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$$

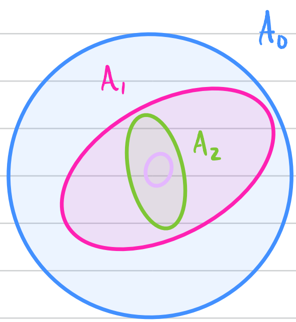
then

$$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \lim_{n\to\infty} \mu(A_n).$$



b) If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of μ-measurable sets with $\mu(A_0) < \infty$, then

$$\mu\Big(\bigcap_{n\in\mathbb{N}} A_n\Big) = \lim_{n\to\infty} \mu(A_n)$$



Caution

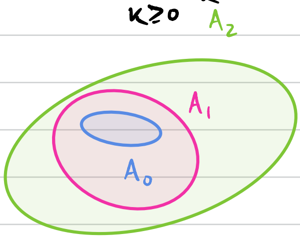
We need $\mu(A_0) < \infty$ for b). Consider e.g. $A_n = (n, \infty)$, $\mu = \lambda$, Lebesgue measure.

Then $\mu(A_n) = \infty$ $\forall n \in \mathbb{N}$ but $\bigcap_{n\geq 1} A_n = \emptyset$.

Proof

a) Disjointify: set $A_0' = A_0$, $A_n' := A_n \setminus A_{n-1}$.

Then $\mu\left(\bigcup_{k\geq 0} A_k\right) = \mu\left(\bigsqcup_{k\geq 0} A_k'\right) = \sum_{k=1}^{\infty} \mu(A_k')$

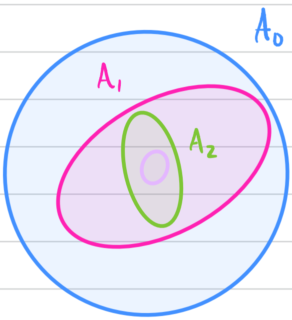


$$= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(A_k')$$

$$= \lim_{n\to\infty} \mu(A_n)$$

b) Set $B_n := A_0 \setminus A_n$, so $\bigcup_{n\geq 0} B_n = A_0 \setminus A_\infty$,

where $A_\infty = \bigcap_{n\geq 0} A_n$.



Then $\{B_n\}_{n\geq 0}$ is an increasing sequence, with

$$\mu\left(\bigcup_{n\geq 0} B_n\right) = \mu(A_0 \setminus A_\infty)$$

$$= \mu(A_0) - \mu(A_\infty)$$

By part a), $\mu\left(\bigcup_{n\geq 0} B_n\right) = \lim_{n\to\infty} \mu(B_n)$

$$= \mu(A_0) - \lim_{n\to\infty} \mu(A_n)$$

Hence, cancelling $\mu(A_0)$ on both sides,

$$\mu(A_\infty) = \lim_{n\to\infty} \mu(A_n).$$

Borel-Cantelli Lemmas

Let P be a property of some points in X. TFAE terminology:

- $\{x \in X : x \text{ satisfies } P\}$ is conull (w.r.t. μ)
- almost every (a.e) point x satisfies P
- P holds almost everywhere (a.e).
- P holds almost surely (a.s.)

We can also say that μ-positively many $x \in X$ satisfy P if $\mu\{x : x \text{ satisfies } P\} > 0$.

Lemma (Borel-Cantelli)
Let $\{A_n\}_{n \geq 0}$ be a sequence of measurable sets

a) If $\sum_{n \geq 0} \mu(A_n) < \infty$ then a.e. $x \in X$ lives in at most finitely many A_n,

i.e the set $\{x \in X : \exists_n^\infty \, x \in A_n\}$

$= \{x \in X : x \text{ lives is infinitely many } A_n\}$ is null.

Lemma (Partial converse / measure compactness)
b) Suppose $\mu(X) < \infty$. If there is a fixed $\varepsilon > 0$ such that $\mu(A_n) \geq \varepsilon$ for all n, then at least an ε-measure set of x lives in

infinitely many A_n's, ie

$$\mu\{x \in X : \exists_n^{\infty}\ x \in A_n\} \geq \varepsilon$$

Proof

a) Put $B = \bigcap_{n \geq 0} \bigcup_{k \geq n} A_n = \limsup_{n \to \infty} A_n$.

Clearly B is measurable. Write $B_n = \bigcup_{k \geq n} A_n$. The $\{B_n\}_{n \geq 0}$ are decreasing, with

$$B = \bigcap_{n \geq 0} B_n$$

Then obviously $B_n \searrow B$, $B \subseteq B_n$ for all $n \in \mathbb{N}$.

Hence, $\mu(B) \leq \mu(B_n)$, and $\lim_{n \to \infty} \mu(B_n) = 0$.

This follows since

$$\mu(B_n) = \mu\left(\bigcup_{k \geq n} A_n\right) \leq \sum_{k \geq n} \mu(A_n)$$

which as a tail of a converging series goes to 0 as $n \to \infty$. Thus, we have shown a).

We have $\mu(B_n) \longrightarrow \mu(B)$ as $n \to \infty$, where

$$B_n = \bigcup_{k \geq n} A_n \text{ again.}$$

Also, for all n, $\mu(B_n) = \mu\left(\bigcup_{k \geq n} A_k\right) \geq \mu(A_n)$

$\geq \varepsilon.$

Hence, $\mu(B) \geq \varepsilon.$ $\square$

Application: Call a sequence $\{A_n\}_{n \geq 0}$ of measurable sets vanishing (resp almost vanishing) if $A_0 \supseteq A_1 \supseteq \ldots$ and $\bigcap_{n \geq 0} A_n$ is empty (resp null).

Let $(X, \mathcal{F}, \mu)$ a finite measure space.

Fact: If $\mathcal{C}$ is a collection of measurable sets closed under countable unions & containing sets of arbitrarily small measure, then there is an almost-vanishing sequence in $\mathcal{C}$.

Proof

For each $n \in \mathbb{N}$, choose A_n with $\mu(A_n) < 2^{-n}$.
Set

$$B_n = \bigcup_{k \geq n} A_k$$

Then $B_n \in \mathcal{C}$, the $\{B_n\}_{n \geq 0}$ are decreasing, and

$$\mu\left(\bigcap_{n \geq 0} B_n\right) = 0 \text{ by Borel-Cantelli,}$$

since $\sum_{n \geq 0} \mu(A_n) \leq \sum_{n \geq 0} 2^{-n} < \infty.$

Measure Exhaustion

Definition

Call a family $\mathcal{C}$ of measurable sets **almost disjoint** if $\forall A \neq B$ in $\mathcal{C}$, $A \cap B$ is null.

Lemma

Let μ be a σ-finite measure. Then there is no uncountable, almost disjoint collection of measurable non-null sets.

Proof

First assume μ is a finite measure. Let $\mathcal{C}$ be an almost disjoint collection of almost disjoint measurable non-null sets. For each $n \in \mathbb{N}$, let $\mathcal{C}_n = \{ C \in \mathcal{C} : \mu(C) \geq \frac{1}{n} \}$. Since every set in $\mathcal{C}$,

$$\mathcal{C} = \bigcup_{n \geq 1} \mathcal{C}_n$$

Clearly each $\mathcal{C}_n$ is finite since $\mu(X) < \infty$, hence $\mathcal{C}$ is countable as a countable union of finite sets. $\lightning$ $\square$

HW: prove for σ-finite measure.

Measure exhaustion lemma

Let μ be a σ-finite measure.

Let $(A_\alpha)_{\alpha<\omega_1}$ be an increasing sequence of measurable sets. Then there is a countable α_∞, ie $\alpha_\infty < \omega_1$, such that $A_{\alpha_\infty} =_\mu A_\alpha$ for all $\alpha > \alpha_\infty$,

ie $\quad \mu(A_\alpha \setminus A_{\alpha_\infty}) = 0$

Here, $=_\mu$ is equality up to a null set, ie $A =_\mu B$ if $A \triangle B$ is null.

Proof

Apply the previous lemma to the family

$$\Big(A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta\Big)_{\alpha<\omega_1}$$

This family is disjoint, hence almost disjoint. Then only countably many of this family have positive measure, ie

$$\Big\{\alpha<\omega_1 : A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta \text{ is non-null}\Big\}$$

is countable, so there exists a supremum $\alpha_\infty < \omega_1$ of that set.

Definition

Fix a measure space $(X, \mathcal{B}, \mu)$. A measurable set A is called an atom if there is no subset $B \subseteq A$ with

$$0 < \mu(B) < \mu(A).$$

μ is called atomless if it has no atoms.

Sierpinski's theorem

In an atomless measure space, μ attains every value in $[0, \mu(X)]$, including if $\mu(X) = \infty$. Ie, for all $r \in [0, \mu(X)]$, there is $A \in \mathcal{B}$ with

$$\mu(A) = r$$

Proof (sketch)

Claim 1: Every measurable X' admits subsets of arbitrarily small measure.

Let $r \in (0, \mu(X))$. Define $r' := \sup\{\mu(B) : \mu(B) \leq r\}$.

Claim 2: the supremum is achieved on a set A. Idea: Baby Zorn's lemma.

Then $r = r'$, since otherwise if $B \subseteq X$ has $\mu(B) = r'$, then take a set $\mathcal{E} \subseteq X \setminus B$ with $\mu(\mathcal{E}) < r - r'$, so $r < \mu(B \cup \mathcal{E}) < r'$, contradicting defn of r'. $\lightning$